THREE-DIMENSIONAL SOLVSOLITONS AND THE MINIMALITY OF THE CORRESPONDING SUBMANIFOLDS

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Abstract. In this paper, we define the corresponding submanifolds to left-invariant Riemannian metrics on Lie groups, and study the following question: does a distinguished left-invariant Riemannian metric on a Lie group correspond to a distinguished submanifold? As a result, we prove that the solvsolitons on three-dimensional simply-connected solvable Lie groups are completely characterized by the minimality of the corresponding submanifolds.

1. Introduction

1.1. **Solvsolitons.** Lie groups with left-invariant Riemannian metrics provide a lot of concrete examples of distinguished Riemannian metrics, such as Einstein metrics and Ricci solitons. Recently, such distinguished left-invariant Riemannian metrics have been studied very actively (see, for instance, [4, 7, 9, 10, 13, 15, 16, 17, 18, 20, 21, 23, 24, 25, 26, 27]).

In this paper, we treat solvsolitons as distinguished left-invariant Riemannian metrics. Recall that a left-invariant Riemannian metric $\langle , \rangle$ on a simply-connected solvable Lie group G is called a *solvsoliton* if the Ricci operator satisfies

$$\mathrm{Ric}_{\langle , \rangle} = cI + D \quad (\text{for some } c \in \mathbb{R} \text{ and } D \in \mathrm{Der}(\mathfrak{g})). \tag{1.1}$$

A solvsoliton on G is called a *nilsoliton* if G is nilpotent. Solvsolitons have been introduced by Lauret ([17]), and play a key role in the study of homogeneous Ricci solitons. In particular, every solvsoliton on a simply-connected solvable Lie group is a Ricci soliton ([17]), and every left-invariant Ricci soliton on a solvable Lie group is isometric to a solvsoliton ([10]).

In the study of solvsolitons, including left-invariant Einstein metrics on solvable Lie groups, the tools from geometric invariant theory have played very important roles. Among others, Lauret ([17]) obtained structural and uniqueness results for solvsolitons. It enables to classify solvsolitons in low-dimensional cases ([17, 27]). For further information, we refer to [15] and references therein.

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1.2. **An approach from the submanifold theory.** In this paper, we propose a new framework for studying distinguished left-invariant Riemannian metrics, such as solvsolitons, in terms of the group actions on and the submanifold theory in noncompact symmetric spaces. This paper only concerns simply-connected solvable Lie groups of dimension three, but here we formulate our framework in a general way.

Let G be a Lie group and $\mathfrak{g}$ be the Lie algebra of G. Consider the set of all left-invariant Riemannian metrics on G, which can be identified with

$$\widetilde{\mathfrak{M}} := \{\langle , \rangle \mid \text{an inner product on } \mathfrak{g}\} \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n), \tag{1.2}$$

where $n = \dim G$. Throughout this paper, this space is assumed to be endowed with the natural $\mathrm{GL}_n(\mathbb{R})$-invariant Riemannian metric (see Subsection 2.1), and hence is a noncompact symmetric space. Let us consider the actions of

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) := \{c\varphi \in \mathrm{GL}_n(\mathbb{R}) \mid c \in \mathbb{R}^\times,\ \varphi \in \mathrm{Aut}(\mathfrak{g})\} \tag{1.3}$$

on $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$. Note that $\mathbb{R}^\times$ denotes the set of nonzero scalar maps on $\mathfrak{g}$, and $\mathrm{Aut}(\mathfrak{g})$ the automorphism group. The group $\mathbb{R}^\times\mathrm{Aut}(\mathfrak{g})$ comes from the equivalence relation "isometry up to scaling" in the Lie algebra level (see Definition 2.1). Denote its equivalence class by [ ]. Then, for each inner product $\langle , \rangle$, it follows from [11] that

$$[\langle , \rangle] = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle, \tag{1.4}$$

which we call *the corresponding submanifold* to $\langle , \rangle$. An important point is that the Riemannian geometric properties of $\langle , \rangle$ are preserved by isometry and scaling. Thus we can regard properties of left-invariant Riemannian metrics as properties of the corresponding submanifolds. Therefore, it would be natural to ask the following:

Question. *Does a distinguished left-invariant Riemannian metric correspond to a distinguished submanifold?*

If an answer for this question is positive, then the approach from the corresponding submanifolds would possibly be useful for the study of left-invariant metrics. For example, the existence and nonexistence problem of distinguished left-invariant Riemannian metrics on G can be translated to the problem of the $\mathbb{R}^\times\mathrm{Aut}(\mathfrak{g})$-action, that is, the existence and nonexistence of distinguished orbits.

1.3. **Results of this paper.** Let G be a three-dimensional simply-connected solvable Lie group with Lie algebra $\mathfrak{g}$. In this paper, we present that there is a good relationship between the existence of solvsolitons on G and geometric aspects of the corresponding action of $\mathbb{R}^\times\mathrm{Aut}(\mathfrak{g})$ on $\mathrm{GL}_3(\mathbb{R})/\mathrm{O}(3)$. We will see this by using the classification of three-dimensional solvable Lie algebras ([2]), which is summarized in Table 1.3. Note that Table 1.3 contains a decomposable one, $\mathfrak{r}_{3,0}$, and our results are true for both decomposable and indecomposable cases.

Name	Non-zero commutation relation	
$\mathfrak{h}_3$	$[e_1, e_2] = e_3$	Nilpotent
$\mathfrak{r}_3$	$[e_1, e_2] = e_2 + e_3,\ [e_1, e_3] = e_3$	Solvable
$\mathfrak{r}_{3,a}$	$[e_1, e_2] = e_2,\ [e_1, e_3] = ae_3 \quad (-1 \leq a \leq 1)$	Solvable
$\mathfrak{r}'_{3,a}$	$[e_1, e_2] = ae_2 - e_3, [e_1, e_3] = e_2 + ae_3 \quad (a \geq 0)$	Solvable

TABLE 1. Three-dimensional solvable Lie algebras

We recall that, for an isometric action on a Riemannian manifold, orbits of maximal dimension are said to be *regular*, and other orbits *singular*. An action is said to be of *cohomogeneity one* if the regular orbits have codimension one. Then, the good relationship we obtain can be summarized as follows.

- Let $\mathfrak{g} = \mathfrak{h}_3$ or $\mathfrak{r}_{3,1}$. Then, $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ acts transitively on $\widetilde{\mathfrak{M}}$, and hence there is the only one orbit. The left-invariant Riemannian metric on G is unique up to isometry and scaling, and the metric is a solvsoliton (nilsoliton for $\mathfrak{h}_3$, and Einstein for $\mathfrak{r}_{3,1}$).
- Let $\mathfrak{g} = \mathfrak{r}_3$. Then, the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is of cohomogeneity one, and all orbits are regular. Furthermore, all orbits are isometrically congruent to each other (namely there are no distinguished orbits). On the other hand, G does not admit a solvsoliton.
- Let $\mathfrak{g} = \mathfrak{r}_{3,a}$ $(-1 \leq a < 1)$. Then, the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is of cohomogeneity one, and all orbits are regular. This action has the unique minimal orbit. On the other hand, G admits a solvsoliton, whose corresponding submanifold coincides with this minimal orbit.
- Let $\mathfrak{g} = \mathfrak{r}'_{3,a}$ $(a \geq 0)$. Then, the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is of cohomogeneity one, and has the unique singular orbit. On the other hand, G admits a left-invariant Einstein metric, whose corresponding submanifold coincides with this singular orbit.

By studying the geometry of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$-orbits in more detail, we obtain a positive answer to the above mentioned Question for three-dimensional solvsolitons. Namely, three-dimensional solvsolitons can completely be characterized by the minimality of the corresponding submanifold.

Main Theorem. *Let G be a three-dimensional simply-connected solvable Lie group, and $\langle , \rangle$ be a left-invariant Riemannian metric on G. Then, $\langle , \rangle$ is a solvsoliton if and only if the corresponding submanifold $[\langle , \rangle]$ is a minimal submanifold in $\widetilde{\mathfrak{M}}$ with respect to the natural $\mathrm{GL}_3(\mathbb{R})$-invariant Riemannian metric.*

This paper is organized as follows. In Section 2, we recall the necessary background on the corresponding submanifolds $[\langle , \rangle]$ to left-invariant Riemannian metrics $\langle , \rangle$ on Lie groups. In Section 3, for each three-dimensional solvable Lie algebra $\mathfrak{g}$, we study the orbit space of the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$. Expressions of the

orbit spaces will be used in both Sections 4 and 5. In Section 4, we study three-dimensional solvsolitons. In particular, we obtain the "Milnor-type theorems" for each $\mathfrak{g}$, and apply them to the reclassification of three-dimensional solvsolitons. In Section 5, we study the actions of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$. The results of Sections 4 and 5 provide the proof of our Main Theorem.

2. The corresponding submanifolds

In this section, we define the notion of the corresponding submanifolds to left-invariant Riemannian metrics on Lie groups. This gives a correspondence between left-invariant Riemannian metrics and $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$-homogeneous submanifolds.

2.1. **The space of left-invariant metrics.** First of all, we recall the space of left-invariant Riemannian metrics, which will be the ambient space of the corresponding submanifolds. We refer to [11].

Let G be an n-dimensional simply-connected Lie group, and $\mathfrak{g}$ be the Lie algebra of G. We consider the set of all left-invariant Riemannian metrics on G, which can naturally be identified with

$$\widetilde{\mathfrak{M}} := \{\langle , \rangle \mid \text{an inner product on } \mathfrak{g}\}. \tag{2.1}$$

We identify $\mathfrak{g}$ with $\mathbb{R}^n$ as vector spaces from now on. Then, since $\mathrm{GL}_n(\mathbb{R})$ acts transitively on $\widetilde{\mathfrak{M}}$ by

$$g.\langle \cdot, \cdot \rangle := \langle g^{-1}(\cdot), g^{-1}(\cdot) \rangle \quad (\text{for } g \in \mathrm{GL}_n(\mathbb{R}),\ \langle , \rangle \in \widetilde{\mathfrak{M}}), \tag{2.2}$$

we have an identification

$$\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n). \tag{2.3}$$

Note that $\widetilde{\mathfrak{M}}$ equipped with the natural $\mathrm{GL}_n(\mathbb{R})$-invariant Riemannian metric is a noncompact Riemannian symmetric space. In order to describe this natural metric, we recall a general theory of reductive homogeneous spaces. Let U/K be a reductive homogeneous space, that is, there exists an Ad_K-invariant subspace $\mathfrak{m}$ of $\mathfrak{u}$ satisfying

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{m}. \tag{2.4}$$

Note that $\mathfrak{u}$ and $\mathfrak{k}$ are the Lie algebras of U and K, respectively, and $\oplus$ is the direct sum as vector spaces. The decomposition (2.4) is called a *reductive decomposition*. Denote by $\pi : U \to U/K$ the natural projection, and by $o := \pi(e)$ the origin of U/K. We identify $\mathfrak{m}$ with the tangent space $\mathrm{T}_o(U/K)$ at o by

$$\mathrm{d}\pi_e|_{\mathfrak{m}} : \mathfrak{m} \to \mathrm{T}_o(U/K). \tag{2.5}$$

This identification induces a one-to-one correspondence between the set of U-invariant Riemannian metrics on U/K and the set of Ad_K-invariant inner products on $\mathfrak{m}$.

Now one can see that $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$ is a reductive homogeneous space, whose reductive decomposition is given by the subspace

$$\mathrm{sym}(n) := \{X \in \mathfrak{gl}_n(\mathbb{R}) \mid X = {}^tX\}. \tag{2.6}$$

We define the $\mathrm{Ad}_{\mathrm{O}(n)}$-inner product on $\mathrm{sym}(n)$ by

$$\langle X, Y \rangle := \mathrm{tr}(XY) \quad (\text{for } X, Y \in \mathrm{sym}(n)). \tag{2.7}$$

We call the $\mathrm{GL}_n(\mathbb{R})$-invariant Riemannian metric corresponding to the above $\mathrm{Ad}_{\mathrm{O}(n)}$-inner product the *natural Riemannian metric*.

2.2. **The corresponding submanifolds.** We now define the submanifolds in the space of left-invariant Riemannian metrics, and see that they are homogeneous. These submanifolds come from the equivalence relation "isometric up to scaling".

Definition 2.1. Two inner products $\langle , \rangle_1$ and $\langle , \rangle_2$ on $\mathfrak{g}$ are said to be *isometric up to scaling* if there exist $k > 0$ and an automorphism $f : \mathfrak{g} \to \mathfrak{g}$ such that $\langle \cdot, \cdot \rangle_1 = k \langle f(\cdot), f(\cdot) \rangle_2$.

Assume that inner products $\langle , \rangle_1$ and $\langle , \rangle_2$ on $\mathfrak{g}$ are isometric up to scaling. Then, the corresponding left-invariant Riemannian metrics on G, the simply-connected Lie group with Lie algebra $\mathfrak{g}$, are isometric up to scaling as Riemannian metrics (we refer to [11, Remark 2.3]). Therefore, this equivalence relation preserves all Riemannian geometric properties of left-invariant metrics. In particular, it preserves solvsolitons.

Definition 2.2. For each inner product $\langle , \rangle$ on $\mathfrak{g}$, we call its equivalence class $[\langle , \rangle]$ the *corresponding submanifold to* $\langle , \rangle$.

Note that $[\langle , \rangle]$ is a submanifold in $\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n)$. We here recall that $[\langle , \rangle]$ is a homogeneous submanifold. Let us denote by

$$\mathbb{R}^\times := \{c \cdot \mathrm{id} : \mathfrak{g} \to \mathfrak{g} \mid c \in \mathbb{R} \setminus \{0\}\}, \tag{2.8}$$

$$\mathrm{Aut}(\mathfrak{g}) := \{\varphi : \mathfrak{g} \to \mathfrak{g} \mid \text{an automorphism}\}. \tag{2.9}$$

Then, the subgroup $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ of $\mathrm{GL}_n(\mathbb{R})$ acts naturally on $\widetilde{\mathfrak{M}}$. Let us denote by $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ the $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$-orbit through $\langle , \rangle$.

Proposition 2.3 ([11, Theorem 2.5])**.** *Let $\langle , \rangle$ be an inner product on $\mathfrak{g}$. Then, the corresponding submanifold $[\langle , \rangle]$ is a homogeneous submanifold with respect to $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$, that is,*

$$[\langle , \rangle] = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle. \tag{2.10}$$

3. Explicit expressions of the moduli spaces

In this section, for each three-dimensional solvable Lie algebra $\mathfrak{g}$, we give an explicit expression of the "moduli space" of left-invariant Riemannian metrics. The results of this section will be used in Sections 4 and 5.

3.1. Preliminaries on the moduli spaces.

In this subsection, we recall some necessary facts on the moduli spaces of left-invariant Riemannian metrics. We refer to [11].

Definition 3.1. For a Lie algebra $\mathfrak{g}$, the quotient space of $\widetilde{\mathfrak{M}}$ by "isometric up to scaling" is called the *moduli space of left-invariant Riemannian metrics*, and denoted by

$$\mathfrak{PM} := \{[\langle , \rangle] \mid \langle , \rangle \in \widetilde{\mathfrak{M}}\}. \tag{3.1}$$

In order to determine $\mathfrak{PM}$ explicitly, we will use the following notion of a set of representatives. Recall that we identify $\mathfrak{g} \cong \mathbb{R}^n$. Denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$, and by $\langle , \rangle_0$ the inner product so that the canonical basis is orthonormal.

Definition 3.2. A subset $U \subset \mathrm{GL}_n(\mathbb{R})$ is called a *set of representatives* of $\mathfrak{PM}$ if it satisfies

$$\mathfrak{PM} = \{[h.\langle , \rangle_0] \mid h \in U\}. \tag{3.2}$$

In the later arguments, it is convenient to use the double cosets. Note that our double coset $[[g]]$ of $g \in \mathrm{GL}_n(\mathbb{R})$ is defined by

$$[[g]] := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \cdot g \cdot \mathrm{O}(n). \tag{3.3}$$

Lemma 3.3 ([6])**.** *Let $U \subset \mathrm{GL}_n(\mathbb{R})$. Then, U is a set of representatives of $\mathfrak{PM}$ if and only if, for every $g \in \mathrm{GL}_n(\mathbb{R})$, there exists $h \in U$ such that $h \in [[g]]$.*

In order to obtain a set of representatives of $\mathfrak{PM}$, one needs $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$. The Lie algebra of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ coincides with $\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g})$, where

$$\mathbb{R} := \{c \cdot \mathrm{id} : \mathfrak{g} \to \mathfrak{g} \mid c \in \mathbb{R}\}, \tag{3.4}$$

$$\mathrm{Der}(\mathfrak{g}) := \{D \in \mathfrak{gl}(\mathfrak{g}) \mid D[\cdot, \cdot] = [D(\cdot), \cdot] + [\cdot, D(\cdot)]\}. \tag{3.5}$$

The Lie algebra $\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g})$ determines $(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0$, the connected component of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ containing the identity.

For each three-dimensional solvable Lie algebra, the moduli space $\mathfrak{PM}$ has been studied in [11]. We here mention the trivial case, which means that $\mathfrak{PM}$ consists of one point.

Proposition 3.4 ([11, 14])**.** *Let $\mathfrak{g} = \mathfrak{h}_3$ or $\mathfrak{r}_{3,1}$. Then, $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ acts transitively on $\widetilde{\mathfrak{M}}$, and hence $\mathfrak{PM} = \{\mathrm{pt}\}$.*

Remark 3.5. One can see that Theorem 1.3 holds for $\mathfrak{g} = \mathfrak{h}_3$ and $\mathfrak{r}_{3,1}$. In fact, it is well-known that any left-invariant Riemannian metrics $\langle , \rangle$ on these Lie algebras are solvsolitons (nilsoliton for $\mathfrak{h}_3$, and Einstein for $\mathfrak{r}_{3,1}$). Furthermore, for every $\langle , \rangle$, the corresponding submanifold $[\langle , \rangle]$ coincides with the ambient space $\widetilde{\mathfrak{M}}$, which is minimal.

In the following, we will study the remaining three-dimensional solvable Lie algebras.

3.2. **A lemma for nontrivial cases.** This subsection gives a preliminary to obtain a set of representatives U of $\mathfrak{PM}$ for $\mathfrak{g} = \mathfrak{r}_3$, $\mathfrak{r}_{3,a}$ $(-1 \leq a < 1)$, and $\mathfrak{r}'_{3,a}$ $(a \geq 0)$.

First of all, let us recall a matrix expression of $\mathrm{Der}(\mathfrak{g})$ for these Lie algebras. The following results can be calculated directly, and be found in [11, Section 4].

Lemma 3.6 ([11]). *The matrix expressions of* $\mathrm{Der}(\mathfrak{g})$ *with respect to the bases* $\{e_1, e_2, e_3\}$ *in Table 1.3 are given as follows*:

(1) *Let* $\mathfrak{g} = \mathfrak{r}_3$. *Then, we have*

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \mid x_{21}, x_{22}, x_{31}, x_{32} \in \mathbb{R} \right\}.$$

(2) *Let* $\mathfrak{g} = \mathfrak{r}_{3,a}$ $(-1 \leq a < 1)$. *Then, we have*

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}.$$

(3) *Let* $\mathfrak{g} = \mathfrak{r}'_{3,a}$ $(a \geq 0)$. *Then, we have*

$$\mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & -x_{23} \\ x_{31} & x_{23} & x_{22} \end{pmatrix} \mid x_{21}, x_{22}, x_{23}, x_{31} \in \mathbb{R} \right\}.$$

Let us consider $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ for these Lie algebras $\mathfrak{g}$. One can see from Lemma 3.6 that $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ contain

$$F := \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{22} \end{pmatrix} \mid x_{11}, x_{22} > 0 \right\}. \tag{3.6}$$

For the later use, we prepare the following lemma, which can be applied for all Lie algebras we have to consider.

Lemma 3.7. *Let* $\mathfrak{g}$ *be a three-dimensional Lie algebra, and fix a basis of* $\mathfrak{g}$. *If* $F \subset \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ *holds, then the following* L' *is a set of representatives of* $\mathfrak{PM}$:

$$L' := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix} \mid a_{33} > 0 \right\}. \tag{3.7}$$

Proof. Take any $g \in \mathrm{GL}_3(\mathbb{R})$. By Lemma 3.3, we have only to show that there exists $g' \in L'$ such that $g' \in [[g]]$. First of all, one knows that there exists $k \in \mathrm{O}(3)$ such that

$$gk = \begin{pmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \quad g_{11}, g_{22}, g_{33} > 0. \tag{3.8}$$

By assumption, we can take

$$\varphi := \frac{1}{g_{11}g_{22}} \begin{pmatrix} g_{22} & 0 & 0 \\ -g_{21} & g_{11} & 0 \\ -g_{31} & 0 & g_{11} \end{pmatrix} \in F \subset \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}). \tag{3.9}$$

By a direct calculation, one has

$$[[g]] \ni \varphi g k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & g_{32}/g_{22} & g_{33}/g_{22} \end{pmatrix} =: g'. \tag{3.10}$$

Since $g' \in L'$, we complete the proof. $\square$

3.3. **Case of $\mathfrak{g} = \mathfrak{r}_3$.** In this subsection, we give an explicit expression of $\mathfrak{PM}$ for $\mathfrak{g} = \mathfrak{r}_3$. We fix a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{r}_3$ whose bracket relations are given by

$$[e_1, e_2] = e_2 + e_3, \quad [e_1, e_3] = e_3. \tag{3.11}$$

From Lemma 3.6, we have

$$\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{31}, x_{32} \in \mathbb{R} \right\}. \tag{3.12}$$

This yields that

$$(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0 = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \mid x_{11}, x_{22} > 0 \right\}. \tag{3.13}$$

Therefore, we can apply Lemma 3.7 for this case.

Proposition 3.8. *Let $\mathfrak{g} = \mathfrak{r}_3$. Then the following U is a set of representatives of $\mathfrak{PM}$:*

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \mid \lambda > 0 \right\}. \tag{3.14}$$

Proof. Take any $g \in \mathrm{GL}_3(\mathbb{R})$. By Lemma 3.3, we have only to show that there exists $\lambda > 0$ such that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \in [[g]]. \tag{3.15}$$

We use L' defined in Lemma 3.7. One has from (3.13) and Lemma 3.7 that there exists $g' \in L'$ such that $g' \in [[g]]$. Since $g' \in L'$, one can write

$$g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}, \quad a_{33} > 0. \tag{3.16}$$

It follows from (3.13) that

$$\varphi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -a_{32} & 1 \end{pmatrix} \in (\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0. \tag{3.17}$$

This shows that

$$[[g]] \ni \varphi g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}. \tag{3.18}$$

Therefore, by putting $\lambda := 1/a_{33}$, we complete the proof. $\square$

3.4. **Case of $\mathfrak{g} = \mathfrak{r}_{3,a}$ $(-1 \leq a < 1)$.** In this subsection, we give an explicit expression of $\mathfrak{PM}$ for $\mathfrak{g} = \mathfrak{r}_{3,a}$. Throughout this subsection, we fix a satisfying $-1 \leq a < 1$, and a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{r}_{3,a}$ whose bracket relations are given by

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = ae_3. \tag{3.19}$$

From Lemma 3.6, we have

$$\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}. \tag{3.20}$$

This yields that

$$(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0 = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{11}, x_{22}, x_{33} > 0 \right\}. \tag{3.21}$$

Proposition 3.9. *Let $\mathfrak{g} = \mathfrak{r}_{3,a}$. Then the following U is a set of representatives of $\mathfrak{PM}$:*

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}. \tag{3.22}$$

Proof. Take any $g \in \mathrm{GL}_3(\mathbb{R})$. By Lemma 3.3, we have only to show that there exists $\lambda \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} \in [[g]]. \tag{3.23}$$

By (3.21) and Lemma 3.7, there exists $g' \in L'$ such that $g' \in [[g]]$. Since $g' \in L'$, one can write

$$g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}, \quad a_{33} > 0. \tag{3.24}$$

It follows from (3.21) that

$$\varphi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a_{33} \end{pmatrix} \in (\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0. \tag{3.25}$$

This yields that

$$[[g]] \ni \varphi g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32}/a_{33} & 1 \end{pmatrix}. \tag{3.26}$$

Therefore, by putting $\lambda := a_{32}/a_{33}$, we complete the proof. $\square$

3.5. **Case of $\mathfrak{g} = \mathfrak{r}'_{3,a}$ ($a \geq 0$).** In this subsection, we give an explicit expression of $\mathfrak{PM}$ for $\mathfrak{g} = \mathfrak{r}'_{3,a}$. Throughout this subsection, we fix a satisfying $a \geq 0$, and a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{r}_{3,a}$ whose bracket relations are given by

$$[e_1, e_2] = ae_2 - e_3, \quad [e_1, e_3] = e_2 + ae_3. \tag{3.27}$$

From Lemma 3.6, we have

$$\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & -x_{23} \\ x_{31} & x_{23} & x_{22} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{23}, x_{31} \in \mathbb{R} \right\}. \tag{3.28}$$

This yields that we can also apply Lemma 3.7 for this case.

Proposition 3.10. *Let* $\mathfrak{g} = \mathfrak{r}'_{3,a}$. *Then the following* U *is a set of representatives of* $\mathfrak{PM}$:

$$U = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \mid \lambda \geq 1 \right\}. \tag{3.29}$$

Proof. Take any $g \in \mathrm{GL}_3(\mathbb{R})$. By Lemma 3.3, we have only to show that there exists $\lambda \geq 1$ such that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \in [[g]]. \tag{3.30}$$

By (3.28), one can see that $(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0$ contains F defined by (3.6). Hence, by Lemma 3.7, there exists $g' \in L'$ such that $g' \in [[g]]$. Since $g' \in L'$, one can write

$$g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}, \ a_{33} > 0. \tag{3.31}$$

Then, from (3.28), one has

$$R(\theta) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \in (\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0. \tag{3.32}$$

It follows from linear algebra (or the theory of Cartan decomposition) that

$$\mathrm{GL}_2(\mathbb{R}) = \mathrm{SO}(2) \cdot \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x \geq y > 0 \right\} \cdot \mathrm{O}(2). \tag{3.33}$$

This yields that there exist $\theta \in \mathbb{R}$ and $k \in \mathrm{O}(3)$ such that

$$[[g]] \ni R(\theta) g' k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix} =: g'', \quad x \geq y > 0 \tag{3.34}$$

By using (3.28) again, one has

$$\varphi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & 1/x \end{pmatrix} \in (\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))^0. \tag{3.35}$$

This yields that

$$[[g]] \ni \varphi g'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y/x \end{pmatrix}. \tag{3.36}$$

Therefore, by putting $\lambda := x/y \geq 1$, we complete the proof. □

4. Three-dimensional solvsolitons

In this section, we give a Milnor-type theorem for each three-dimensional solvable Lie algebra $\mathfrak{g}$, and apply it to determine which points in the moduli space $\mathfrak{PM}$ are solvsolitons. Note that a classification of three-dimensional solvsolitons has already been obtained by Lauret ([17]), but we here reprove it, since Milnor-type theorems itself and their application would be interesting.

4.1. **Preliminaries on curvatures.** In this subsection, we recall the notion of solvsolitons introduced by Lauret ([17]), and study the Ricci operators of three-dimensional solvable Lie algebras. Note that we discuss everything on a metric Lie algebra $(\mathfrak{g}, \langle , \rangle)$, instead of the simply-connected Lie group with Lie algebra $\mathfrak{g}$ equipped with the corresponding left-invariant Riemannian metric.

Definition 4.1. An inner product $\langle , \rangle$ on a solvable Lie algebra $\mathfrak{g}$ is called a *solvsoliton* if it satisfies

$$\mathrm{Ric}_{\langle , \rangle} \in \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}), \tag{4.1}$$

where $\mathrm{Ric}_{\langle , \rangle}$ is the Ricci operator of $\langle , \rangle$. If $\mathfrak{g}$ is nilpotent, then a solvsoliton on $\mathfrak{g}$ is called a *nilsoliton*.

Here we recall the definition of the Ricci operator of $(\mathfrak{g}, \langle , \rangle)$. First of all, the Levi-Civita connection $\nabla : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is given by

$$2\langle \nabla_X Y, Z\rangle = \langle [Z, X], Y\rangle + \langle X, [Z, Y]\rangle + \langle [X, Y], Z\rangle. \tag{4.2}$$

The Riemannian curvature R is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{4.3}$$

Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $\langle , \rangle$. The Ricci operator $\mathrm{Ric}_{\langle , \rangle} : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\mathrm{Ric}_{\langle , \rangle}(X) := \sum R(X, e_i)e_i. \tag{4.4}$$

Let us consider the equivalence relation, isometry and scaling in the sense of Definition 2.1. Recall that $[\langle , \rangle]$ denotes the equivalence class of $\langle , \rangle$. Then it is easy to see the following.

Proposition 4.2. *Let $\langle , \rangle$ and $\langle , \rangle'$ be inner products on a solvable Lie algebra $\mathfrak{g}$, and assume that $[\langle , \rangle] = [\langle , \rangle']$. If $\langle , \rangle$ is a solvsoliton, then so is $\langle , \rangle'$.*

This proposition is an easy observation, but has an important conclusion. That is, it is enough to consider $\mathfrak{PM}$ to examine whether $\mathfrak{g}$ admits a solvsoliton or not.

Remark 4.3. It is worthwhile to mention that the uniqueness of solvsolitons holds. That is, if $\langle , \rangle$ and $\langle , \rangle'$ are solvsolitons on a solvable Lie algebra $\mathfrak{g}$, then $[\langle , \rangle] = [\langle , \rangle']$ holds. This follows from the proof of [17, Theorem 5.1]. But, we will not use this in the latter arguments. In particular, for solvsolitons on three-dimensional solvable Lie algebras, the uniqueness can be directly seen from our classification.

At the end of this subsection, we calculate the Ricci curvatures of three-dimensional solvable Lie algebras in a unified way.

Lemma 4.4. *Let $\mathfrak{g}$ be a three-dimensional solvable Lie algebra, and $\langle , \rangle$ be an inner product on $\mathfrak{g}$. Suppose that there exist $a, b, c, d \in \mathbb{R}$ and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $\langle , \rangle$ such that the bracket relations are given by*

$$[x_1, x_2] = ax_2 + bx_3, \quad [x_1, x_3] = cx_2 + dx_3.$$

Then, the Ricci operator satisfies

$$\mathrm{Ric}_{\langle , \rangle}(x_i) = \begin{cases} -(a^2 + d^2 + (1/2)(b + c)^2)\, x_1 & (i = 1), \\ -(a(a + d) + (1/2)(b^2 - c^2))\, x_2 - (ac + bd)\, x_3 & (i = 2), \\ -(ac + bd)\, x_2 - (d(a + d) - (1/2)(b^2 - c^2))\, x_3 & (i = 3). \end{cases}$$

Proof. First of all, we calculate the Levi-Civita connection ∇. A direct calculation shows that

$$\nabla_{x_1} x_1 = 0, \quad \nabla_{x_2} x_2 = ax_1, \quad \nabla_{x_3} x_3 = dx_1. \tag{4.5}$$

In order to calculate the other components, we use $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$2\langle U(X, Y), Z\rangle = \langle [Z, X], Y\rangle + \langle X, [Z, Y]\rangle$$

for every $X, Y, Z \in \mathfrak{g}$. One can easily calculate that

$$U(x_1, x_2) = -(a/2)x_2 - (c/2)x_3. \tag{4.6}$$

Note that U is symmetric. Hence, one obtains that

$$\begin{aligned} \nabla_{x_1} x_2 &= (1/2)[x_1, x_2] + U(x_1, x_2) = ((b - c)/2)x_3, \\ \nabla_{x_2} x_1 &= (1/2)[x_2, x_1] + U(x_2, x_1) = -ax_2 - ((b + c)/2)x_3. \end{aligned} \tag{4.7}$$

By changing the roles of x_2 and x_3, we also have

$$\nabla_{x_1} x_3 = ((c - b)/2)x_2, \quad \nabla_{x_3} x_1 = -dx_3 - ((b + c)/2)x_2. \tag{4.8}$$

A similar calculation shows that $U(x_2, x_3) = ((b + c)/2)x_1$, which concludes

$$\nabla_{x_2} x_3 = ((b + c)/2)x_1, \quad \nabla_{x_3} x_2 = ((b + c)/2)x_1. \tag{4.9}$$

One can thus calculate the Riemannian curvatures R. The above calculations of ∇ yield that

$$\begin{aligned} R(x_1, x_2)x_2 &= -(a^2 + (3/4)b^2 - (1/4)c^2 + (1/2)bc)x_1, \\ R(x_1, x_3)x_3 &= -(-(1/4)b^2 + (3/4)c^2 + d^2 + (1/2)bc)x_1. \end{aligned}$$

By summing up them, we obtain the Ricci curvature $\mathrm{Ric}_{\langle,\rangle}(x_1)$. Similarly, one can obtain $\mathrm{Ric}_{\langle,\rangle}(x_2)$ and $\mathrm{Ric}_{\langle,\rangle}(x_3)$ by

$$\begin{aligned} R(x_2,x_1)x_1 &= -(a^2+(3/4)b^2-(1/4)c^2+(1/2)bc)x_2-(ac+bd)x_3, \\ R(x_2,x_3)x_3 &= ((1/4)b^2+(1/4)c^2-ad+(1/2)bc)x_2, \\ R(x_3,x_1)x_1 &= -(ac+bd)x_2-(-(1/4)b^2+(3/4)c^2+d^2+(1/2)bc)x_3, \\ R(x_3,x_2)x_2 &= ((1/4)b^2+(1/4)c^2-ad+(1/2)bc)x_3. \end{aligned}$$

This completes the proof of the lemma. $\square$

Lemma 4.4 is a slight generalization of some known results. In fact, when $a+d \neq 0$ and $ac+bd=0$, the Ricci operators were calculated by Milnor ([19, Lemma 6.5]). Note that the Ricci operators are diagonal in this case. Ha and Lee ([5]) also calculated the Ricci operators in some cases, which essentially correspond to the case of $a=0$.

4.2. **Preliminaries on Milnor-type theorems.** In this subsection, we recall a method for studying all inner products on a given Lie algebra $\mathfrak{g}$. This method is called a Milnor-type theorem in [6], since it generalizes the famous theorem by Milnor ([19]).

Theorem 4.5. *Let U be a set of representatives of $\mathfrak{PM}$. Then, for every inner product $\langle,\rangle$ on $\mathfrak{g}$, we have the following*:

(1) *There exist $h \in U$, $\varphi \in \mathrm{Aut}(\mathfrak{g})$, and $k>0$ such that $\{\varphi h e_1, \ldots, \varphi h e_n\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $k\langle,\rangle$.*
(2) *The matrix expression of $\mathrm{Der}(\mathfrak{g})$ with respect to $\{\varphi h e_1, \ldots, \varphi h e_n\}$ coincides with*

$$\{h^{-1}Dh \in \mathrm{GL}_n(\mathbb{R}) \mid D \in \mathrm{Der}(\mathfrak{g})\}.$$

Proof. The first assertion has been proved in [6]. We show the second assertion. One has that $\{\varphi h e_1, \ldots, \varphi h e_n\}$ and $\{h e_1, \ldots, h e_n\}$ have the same bracket relations, since $\varphi \in \mathrm{Aut}(\mathfrak{g})$. This yields that the matrix expressions of $\mathrm{Der}(\mathfrak{g})$ with respect to these two bases are the same. Furthermore, the latter basis and $\{e_1, \ldots, e_n\}$ are related by

$$(he_1, \ldots, he_n) = (e_1, \ldots, e_n)h. \tag{4.10}$$

Therefore, an elementary linear algebra shows that the matrix expression of $\mathrm{Der}(\mathfrak{g})$ with respect to $\{he_1, \ldots, he_n\}$ coincides with the one in the second assertion. This completes the proof. $\square$

By applying this theorem for a given Lie algebra $\mathfrak{g}$, we can obtain a Milnor-type theorem. More precisely, the basis $\{\varphi h e_1, \ldots, \varphi h e_n\}$ plays a similar role to the Milnor frames. Note that the bracket relations among elements of this basis depend only on $h \in U$, since φ preserves the bracket product.

In the following subsections, we will study the existence of solvsolitons on three-dimensional solvable Lie algebras. Note that we can omit the cases of $\mathfrak{g} = \mathfrak{h}_3$ and $\mathfrak{r}_{3,1}$, because of Remark 3.5.

4.3. **Case of $\mathfrak{g} = \mathfrak{r}_3$.** In this subsection, we prove that $\mathfrak{g} = \mathfrak{r}_3$ does not admit solvsolitons. The main tool is the following Milnor-type theorem.

Proposition 4.6. *For every inner product $\langle , \rangle$ on $\mathfrak{g} = \mathfrak{r}_3$, there exist $\lambda > 0$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle , \rangle$ such that the bracket relations are given by*

$$[x_1, x_2] = x_2 + \lambda x_3, \quad [x_1, x_3] = x_3. \tag{4.11}$$

Furthermore, the matrix expression of $\mathrm{Der}(\mathfrak{g})$ *with respect to* $\{x_1, x_2, x_3\}$ *coincides with*

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \mid x_{21}, x_{22}, x_{31}, x_{32} \in \mathbb{R} \right\}.$$

Proof. Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathfrak{r}_3$. Recall that the bracket relations are given by

$$[e_1, e_2] = e_2 + e_3, \quad [e_1, e_3] = e_3. \tag{4.12}$$

We have proved in Proposition 3.8 that the following U is a set of representatives of $\mathfrak{PM}$:

$$U := \left\{ g_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \mid \lambda > 0 \right\}. \tag{4.13}$$

Take any inner product $\langle , \rangle$ on $\mathfrak{g}$. By Theorem 4.5, there exist $g_\lambda \in U$, $k > 0$, and $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\{\varphi g_\lambda e_1, \varphi g_\lambda e_2, \varphi g_\lambda e_3\}$ is orthonormal with respect to $k\langle , \rangle$. Put $x_i := \varphi g_\lambda e_i$ for $i = 1, 2, 3$. We calculate the bracket relations among them. One has

$$g_\lambda e_1 = e_1, \quad g_\lambda e_2 = e_2, \quad g_\lambda e_3 = (1/\lambda) e_3. \tag{4.14}$$

We thus obtain

$$\begin{aligned} [g_\lambda e_1, g_\lambda e_2] &= [e_1, e_2] = e_2 + e_3 = g_\lambda e_2 + \lambda g_\lambda e_3, \\ [g_\lambda e_1, g_\lambda e_3] &= [e_1, (1/\lambda) e_3] = (1/\lambda) e_3 = g_\lambda e_3, \\ [g_\lambda e_2, g_\lambda e_3] &= [e_2, (1/\lambda) e_3] = 0. \end{aligned} \tag{4.15}$$

Therefore, by applying $\varphi \in \mathrm{Aut}(\mathfrak{g})$ to the both sides of these equations, we obtain

$$\begin{aligned} [x_1, x_2] &= [\varphi g_\lambda e_1, \varphi g_\lambda e_2] = \varphi[g_\lambda e_1, g_\lambda e_2] = x_2 + \lambda x_3, \\ [x_1, x_3] &= [\varphi g_\lambda e_1, \varphi g_\lambda e_3] = \varphi[g_\lambda e_1, g_\lambda e_3] = x_3, \\ [x_2, x_3] &= [\varphi g_\lambda e_2, \varphi g_\lambda e_3] = \varphi[g_\lambda e_2, g_\lambda e_3] = 0. \end{aligned} \tag{4.16}$$

This completes the proof of the first assertion. We show the second assertion. Lemma 3.6 yields that, for every $D \in \mathrm{Der}(\mathfrak{g})$, the matrix expression of D with respect to $\{e_1, e_2, e_3\}$ is given by

$$D = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix}. \tag{4.17}$$

A direct calculation shows that

$$g_\lambda^{-1} \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} g_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ \lambda x_{31} & \lambda x_{32} & x_{22} \end{pmatrix}. \tag{4.18}$$

Note that λx_{31} and λx_{32} can take any real numbers, and are independent of the other components. Therefore, by Theorem 4.5 (2), one can obtain the matrix expression of $\mathrm{Der}(\mathfrak{g})$ with respect to $\{x_1, x_2, x_3\}$. This completes the proof of the second assertion. □

By applying the Milnor-type theorem, Proposition 4.6, we prove that $\mathfrak{r}_3$ does not admit solvsolitons.

Proposition 4.7. *The Lie algebra $\mathfrak{g} = \mathfrak{r}_3$ does not admit solvsolitons.*

Proof. Take any inner product $\langle , \rangle$ on $\mathfrak{g}$. We show that this is not a solvsoliton. By Proposition 4.6, there exist $\lambda > 0$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle , \rangle$ such that the bracket relations are given by

$$[x_1, x_2] = x_2 + \lambda x_3, \quad [x_1, x_3] = x_3. \tag{4.19}$$

We can assume $k = 1$ without loss of generality, since solvsolitons are preserved by scaling. Then, from Lemma 4.4, the matrix expression of $\mathrm{Ric}_{\langle , \rangle}$ with respect to the orthonormal basis $\{x_1, x_2, x_3\}$ is given by

$$\mathrm{Ric}_{\langle , \rangle} = - \begin{pmatrix} 2 + (\lambda^2/2) & 0 & \\ 0 & 2 + (\lambda^2/2) & \lambda \\ 0 & \lambda & 2 - (\lambda^2/2) \end{pmatrix}. \tag{4.20}$$

On the other hand, by Proposition 4.6, one knows the matrix expression of $\mathrm{Der}(\mathfrak{g})$ with respect to $\{x_1, x_2, x_3\}$. By looking at the $(2, 3)$-component, we have

$$\mathrm{Ric}_{\langle , \rangle} \notin \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}). \tag{4.21}$$

This proves that $\langle , \rangle$ is not a solvsoliton. □

4.4. **Case of $\mathfrak{g} = \mathfrak{r}_{3,a}$ $(-1 \leq a < 1)$.** In this subsection, we classify solvsolitons on $\mathfrak{g} = \mathfrak{r}_{3,a}$. Throughout this subsection, we fix a satisfying $-1 \leq a < 1$. Recall that, for the canonical basis $\{e_1, e_2, e_3\}$ of $\mathfrak{r}_{3,a}$, the bracket relations are given by

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = ae_3. \tag{4.22}$$

Proposition 4.8. *For every inner product $\langle , \rangle$ on $\mathfrak{g} = \mathfrak{r}_{3,a}$, there exist $\lambda \in \mathbb{R}$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle , \rangle$ such that the bracket relations are given by*

$$[x_1, x_2] = x_2 + \lambda(a-1)x_3, \quad [x_1, x_3] = ax_3.$$

Furthermore, the matrix expression of $\mathrm{Der}(\mathfrak{g})$ *with respect to* $\{x_1, x_2, x_3\}$ *coincides with*

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & \lambda(x_{33} - x_{22}) & x_{33} \end{pmatrix} \mid x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}.$$

Proof. The proof is similar to that of Proposition 4.6. Take any inner product $\langle , \rangle$ on $\mathfrak{r}_{3,a}$. By Proposition 3.9, the following U is a set of representatives of $\mathfrak{PM}$:

$$U := \left\{ g_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} \mid \lambda \in \mathbb{R} \right\}. \tag{4.23}$$

By Theorem 4.5, there exist $g_\lambda \in U$, $k > 0$, and $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that

$$(x_1, x_2, x_3) := (\varphi g_\lambda e_1, \varphi g_\lambda e_2, \varphi g_\lambda e_3) \tag{4.24}$$

forms an orthonormal basis with respect to $k\langle , \rangle$. We have only to check the bracket relations. By definition, we have

$$g_\lambda e_1 = e_1, \quad g_\lambda e_2 = e_2 + \lambda e_3, \quad g_\lambda e_3 = e_3. \tag{4.25}$$

One can thus calculate that

$$\begin{aligned} [g_\lambda e_1, g_\lambda e_2] &= [e_1, e_2 + \lambda e_3] = e_2 + a\lambda e_3 = (g_\lambda e_2 - \lambda g_\lambda e_3) + a\lambda g_\lambda e_3 \\ &\qquad\qquad = g_\lambda e_2 + \lambda(a-1)e_3, \\ [g_\lambda e_1, g_\lambda e_3] &= [e_1, e_3] = ae_3 = ag_\lambda e_3, \\ [g_\lambda e_2, g_\lambda e_3] &= [e_2 + \lambda e_3, e_3] = 0. \end{aligned} \tag{4.26}$$

By applying $\varphi \in \mathrm{Aut}(\mathfrak{g})$, one completes the proof of the first assertion. The second assertion follows from Lemma 3.6 and Theorem 4.5. In fact, one has

$$g_\lambda^{-1} \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} g_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ -\lambda x_{21} + x_{31} & \lambda(x_{33} - x_{22}) & x_{33} \end{pmatrix}. \tag{4.27}$$

This completes the proof, since $-\lambda x_{21} + x_{31}$ can take any real number and is independent of the other components. □

By applying the Milnor-type theorem, Proposition 4.8, one can classify solvsolitons on $\mathfrak{g} = \mathfrak{r}_{3,a}$. Recall that $\langle , \rangle_0$ is the inner product on $\mathfrak{g}$ so that the canonical basis $\{e_1, e_2, e_3\}$ is orthonormal.

Proposition 4.9. *An inner product $\langle , \rangle$ on $\mathfrak{g} = \mathfrak{r}_{3,a}$ is a solvsoliton if and only if* $[\langle , \rangle] = [\langle , \rangle_0]$.

Proof. First of all, we show the "if"-part. We have only to show that $\langle , \rangle_0$ is a solvsoliton. By Lemma 4.4, one knows

$$\mathrm{Ric}_{\langle,\rangle_0} = -\begin{pmatrix} 1+a^2 & 0 & 0 \\ 0 & 1+a & 0 \\ 0 & 0 & a(1+a) \end{pmatrix}, \tag{4.28}$$

One also knows by Lemma 3.6 that

$$\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}. \tag{4.29}$$

Then we have $\mathrm{Ric}_{\langle,\rangle_0} \in \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g})$, that is, $\langle , \rangle_0$ is a solvsoliton.

We show the "only if"-part. Take any inner product $\langle , \rangle$ on $\mathfrak{g} = \mathfrak{r}_{3,a}$, and assume that it is a solvsoliton. Proposition 4.8 yields that there exist $\lambda \in \mathbb{R}$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle , \rangle$ such that the bracket relations are given by

$$[x_1, x_2] = x_2 + \lambda(a-1)x_3, \quad [x_1, x_3] = ax_3. \tag{4.30}$$

We can assume $k = 1$ without loss of generality. Hence $\{x_1, x_2, x_3\}$ is orthonormal. For simplicity of the notation, we put

$$T := (1/2)\lambda^2(a-1)^2. \tag{4.31}$$

Then, from Lemma 4.4, one obtains the matrix expressions of $\mathrm{Ric}_{\langle,\rangle}$ with respect to the basis $\{x_1, x_2, x_3\}$ as follows:

$$\mathrm{Ric}_{\langle,\rangle} = -\begin{pmatrix} 1+a^2+T & 0 & 0 \\ 0 & 1+a+T & \lambda a(a-1) \\ 0 & \lambda a(a-1) & a+a^2-T \end{pmatrix}. \tag{4.32}$$

On the other hand, Proposition 4.8 gives the matrix expression with respect to $\{x_1, x_2, x_3\}$ as follows:

$$\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & \lambda(x_{33} - x_{22}) & x_{33} \end{pmatrix} \right\}. \tag{4.33}$$

We here claim that $\lambda = 0$. Recall that $\langle , \rangle$ is a solvsoliton. Hence, by looking at the $(2,3)$-component, we have

$$\lambda a(a-1) = 0. \tag{4.34}$$

Assume that $\lambda \neq 0$. Since $-1 \leq a < 1$, one has $a = 0$. Then, by looking at the $(3,2)$-component, we have

$$0 = \lambda(-T - (1+T)) = \lambda(-1-\lambda^2) \neq 0. \tag{4.35}$$

This is a contradiction, which shows the claim.

Since $\lambda = 0$, one can see that $\{e_1, e_2, e_3\}$ and $\{x_1, x_2, x_3\}$ have the same bracket relations. Thus, a linear map $F : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$F(e_i) = x_i \quad (i = 1, 2, 3) \tag{4.36}$$

gives an isometry from $(\mathfrak{g}, \langle , \rangle_0)$ onto $(\mathfrak{g}, \langle , \rangle)$. This proves $[\langle , \rangle] = [\langle , \rangle_0]$. $\square$

4.5. **Case of $\mathfrak{g} = \mathfrak{r}'_{3,a}$ ($a \geq 0$).** In this subsection, we classify solvsolitons on $\mathfrak{g} = \mathfrak{r}'_{3,a}$. Throughout this subsection, we fix a satisfying $a \geq 0$. Recall that, for the canonical basis $\{e_1, e_2, e_3\}$, the bracket relations are given by

$$[e_1, e_2] = ae_2 - e_3, \quad [e_1, e_3] = e_2 + ae_3. \tag{4.37}$$

Proposition 4.10. *For every inner product $\langle , \rangle$ on $\mathfrak{g} = \mathfrak{r}'_{3,a}$, there exist $\lambda \geq 1$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle , \rangle$ such that the bracket relations are given by*

$$[x_1, x_2] = ax_2 - \lambda x_3, \quad [x_1, x_3] = (1/\lambda)x_2 + ax_3. \tag{4.38}$$

Furthermore, the matrix expression of $\mathrm{Der}(\mathfrak{g})$ *with respect to $\{x_1, x_2, x_3\}$ coincides with*

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ x_{31} & -\lambda^2 x_{23} & x_{22} \end{pmatrix} \mid x_{21}, x_{22}, x_{23}, x_{31} \in \mathbb{R} \right\}.$$

Proof. The proof is similar to that of Proposition 4.6. Take any inner product $\langle , \rangle$ on $\mathfrak{r}'_{3,a}$. By Proposition 3.10, the following U is a set of representatives of $\mathfrak{PM}$:

$$U := \left\{ g_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \mid \lambda \geq 1 \right\}. \tag{4.39}$$

By Theorem 4.5, there exist $g_\lambda \in U$, $k > 0$, and $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that

$$(x_1, x_2, x_3) := (\varphi g_\lambda e_1, \varphi g_\lambda e_2, \varphi g_\lambda e_3) \tag{4.40}$$

forms an orthonormal basis with respect to $k\langle , \rangle$. We have only to check the bracket relations. By definition, we have

$$g_\lambda e_1 = e_1, \quad g_\lambda e_2 = e_2, \quad g_\lambda e_3 = (1/\lambda)e_3. \tag{4.41}$$

One can thus calculate that

$$\begin{aligned} &[g_\lambda e_1, g_\lambda e_2] = [e_1, e_2] = ae_2 - e_3 = ag_\lambda e_2 - \lambda g_\lambda e_3, \\ &[g_\lambda e_1, g_\lambda e_3] = [e_1, (1/\lambda)e_3] = (1/\lambda)(e_2 + ae_3) = (1/\lambda)g_\lambda e_2 + ag_\lambda e_3, \\ &[g_\lambda e_2, g_\lambda e_3] = [e_2, (1/\lambda)e_3] = 0. \end{aligned} \tag{4.42}$$

By applying $\varphi \in \mathrm{Aut}(\mathfrak{g})$, one completes the proof of the first assertion. The second assertion follows from Lemma 3.6 and Theorem 4.5. In fact, one has

$$g_\lambda^{-1} \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & -x_{23} \\ x_{31} & x_{23} & x_{22} \end{pmatrix} g_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & x_{22} & -(1/\lambda)x_{23} \\ \lambda x_{31} & \lambda x_{23} & x_{22} \end{pmatrix}. \tag{4.43}$$

This completes the proof by changing λx_{31} to x_{31}, and $-(1/\lambda)x_{23}$ to x_{23}. $\square$

By applying the Milnor-type theorem, Proposition 4.10, one can classify solvsolitons on $\mathfrak{g} = \mathfrak{r}'_{3,a}$. In fact, this admits a left-invariant Einstein metric. Recall that $\langle , \rangle_0$ is the inner product so that the canonical basis $\{e_1, e_2, e_3\}$ is orthonormal.

Proposition 4.11. *An inner product $\langle , \rangle$ on $\mathfrak{g} = \mathfrak{r}'_{3,a}$ is a solvsoliton if and only if $[\langle , \rangle] = [\langle , \rangle_0]$. In fact, $\langle , \rangle_0$ is Einstein.*

Proof. The proof is similar to that of Proposition 4.9. First of all, we show the "if"-part. By Lemma 4.4, one knows

$$\mathrm{Ric}_{\langle , \rangle_0} = - \begin{pmatrix} 2a^2 & 0 & 0 \\ 0 & 2a^2 & 0 \\ 0 & 0 & 2a^2 \end{pmatrix}. \tag{4.44}$$

This shows that $\langle , \rangle_0$ is Einstein, and hence a solvsoliton.

We show the "only if"-part. Take any inner product $\langle , \rangle$ on $\mathfrak{g} = \mathfrak{r}'_{3,a}$, and assume that it is a solvsoliton. Proposition 4.10 yields that there exist $\lambda \geq 1$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle , \rangle$ such that the bracket relations are given by

$$[x_1, x_2] = ax_2 - \lambda x_3, \quad [x_1, x_3] = (1/\lambda)x_2 + ax_3. \tag{4.45}$$

We can assume $k = 1$ without loss of generality. Hence $\{x_1, x_2, x_3\}$ is orthonormal. For simplicity of the notation, we put

$$S := \lambda - (1/\lambda). \tag{4.46}$$

Then, from Lemma 4.4, one obtains the matrix expressions of $\mathrm{Ric}_{\langle , \rangle}$ with respect to the basis $\{x_1, x_2, x_3\}$ as follows:

$$\mathrm{Ric}_{\langle , \rangle} = -\frac{1}{2} \begin{pmatrix} 4a^2 + S^2 & 0 & 0 \\ 0 & 4a^2 + (\lambda^2 - (1/\lambda)^2) & -2aS \\ 0 & -2aS & 4a^2 - (\lambda^2 - (1/\lambda)^2) \end{pmatrix}. \tag{4.47}$$

On the other hand, Proposition 4.10 gives the matrix expression with respect to $\{x_1, x_2, x_3\}$ as follows:

$$\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ x_{31} & -\lambda^2 x_{23} & x_{22} \end{pmatrix} \right\}. \tag{4.48}$$

We here show that $\lambda = 1$. Recall that $\langle , \rangle$ is a solvsoliton. Hence, By looking at the $(2,2)$ and $(3,3)$-components, we have

$$4a^2 + (\lambda^2 - (1/\lambda)^2) = 4a^2 - (\lambda^2 - (1/\lambda)^2). \tag{4.49}$$

Since $\lambda \geq 1$, this yields that

$$\lambda = 1. \tag{4.50}$$

Since $\lambda = 1$, one can see that $\{e_1, e_2, e_3\}$ and $\{x_1, x_2, x_3\}$ have the same bracket relations. Thus, a linear map $F : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$F(e_i) = x_i \quad (i = 1, 2, 3) \tag{4.51}$$

gives an isometry from $(\mathfrak{g}, \langle , \rangle_0)$ onto $(\mathfrak{g}, \langle , \rangle)$. This proves $[\langle , \rangle] = [\langle , \rangle_0]$. □

5. The minimality of the corresponding submanifolds

In this section, we study the actions of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ and examine the minimality of its orbits, the corresponding submanifolds to left-invariant metrics. After some necessary preliminaries in Subsection 5.1, we study the cases of $\mathfrak{g} = \mathfrak{r}_3$, $\mathfrak{r}_{3,a}$ $(-1 \leq a < 1)$, and $\mathfrak{r}'_{3,a}$ $(a \geq 0)$ in Subsections 5.2, 5.3, and 5.4, respectively. We have only to study these cases, since the actions of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is transitive for the remaining cases $\mathfrak{g} = \mathfrak{h}_3$ and $\mathfrak{r}_{3,1}$.

5.1. Preliminary.

In this subsection, we review some of the standard facts on reductive homogeneous spaces and homogeneous submanifolds. We refer to [1, 3].

Let U/K be a reductive homogeneous space with a reductive decomposition

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{m}. \tag{5.1}$$

As in Subsection 2.1, denote by $\pi : U \to U/K$ the natural projection, and by $o := \pi(e)$ the origin of U/K. We identify $\mathfrak{m}$ with the tangent space $\mathrm{T}_o(U/K)$ at o by

$$\mathrm{d}\pi_e|_{\mathfrak{m}} : \mathfrak{m} \to \mathrm{T}_o(U/K). \tag{5.2}$$

In the following, we equip a U-invariant Riemannian metric g on U/K.

We here recall a formula for the Levi-Civita connection ∇ of g. For any $X \in \mathfrak{u}$, we define the fundamental vector field X^* on U/K by

$$X^*_p = \frac{d}{dt}(\exp tX).p|_{t=0} \quad (\text{for } p \in U/K). \tag{5.3}$$

Let $X, Y, Z \in \mathfrak{u}$. Then one knows

$$X^*_o = \mathrm{d}\pi_e(X), \tag{5.4}$$

$$[X^*, Y^*] = -[X, Y]^*, \tag{5.5}$$

$$2g(\nabla_{X^*} Y^*, Z^*) = g([X^*, Y^*], Z^*) + g([X^*, Z^*], Y^*) + g(X^*, [Y^*, Z^*]). \tag{5.6}$$

We now consider homogeneous submanifolds in $(U/K, g)$. Let U' be a Lie subgroup of U, and consider the orbit $U'.o$ through the origin o. Let $\mathfrak{u}'$ be the

Lie algebra of U', and denote by $\langle , \rangle$ the inner product on $\mathfrak{m}$ corresponding to g. We define

$$\mathfrak{m}' := \mathrm{d}\pi_e(\mathfrak{u}') \cong \mathrm{T}_o(U'.o). \tag{5.7}$$

Denote by $\mathfrak{m} \ominus \mathfrak{m}'$ the orthogonal complement of $\mathfrak{m}'$ in $\mathfrak{m}$ with respect to $\langle , \rangle$. Then, the second fundamental form $h : \mathfrak{m}' \times \mathfrak{m}' \to \mathfrak{m} \ominus \mathfrak{m}'$ of $U'.o$ at o is defined by

$$h(X_o^*, Y_o^*) := (\nabla_{X^*} Y^* - \nabla'_{X^*} Y^*)_o \quad (\text{for } X, Y \in \mathfrak{u}'), \tag{5.8}$$

where ∇' is the Levi-Civita connection of $U'.o$ with respect to the induced metric. Take $Z \in \mathfrak{u}$ satisfying $Z_o^* \in \mathfrak{m} \ominus \mathfrak{m}'$. From (5.5) and (5.6), one obtains

$$2\langle h(X_o^*, Y_o^*), Z_o^* \rangle = \langle [Z, X]_o^*, Y_o^* \rangle + \langle X_o^*, [Z, Y]_o^* \rangle. \tag{5.9}$$

The mean curvature vector of $U'.o$ at o is defined by

$$H := -(1/k)\mathrm{tr}(h) = -(1/k) \sum h(E_i', E_i'), \tag{5.10}$$

where $\{E_i'\}$ is an orthonormal basis of $\mathfrak{m}'$, and k is the dimension of $U'.o$. We call $U'.o$ *minimal* if its mean curvature vector is equal to zero.

In the following subsections, we will calculate the mean curvature vectors of the corresponding submanifolds in $\mathrm{GL}_3(\mathbb{R})/\mathrm{O}(3)$ with respect to the natural Riemannian metric (see Section 2). We will frequently use

$$\mathrm{d}\pi_e : \mathfrak{gl}_3(\mathrm{R}) \to \mathrm{sym}(3) : X \mapsto (1/2)(X + {}^tX). \tag{5.11}$$

5.2. **Case of $\mathfrak{g} = \mathfrak{r}_3$.** In this subsection, we study the case of $\mathfrak{g} = \mathfrak{r}_3$. First of all, by direct calculations, one has

$$\mathrm{Aut}(\mathfrak{g}) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \mid x_{22} \neq 0 \right\}. \tag{5.12}$$

This easily yields that

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \mid x_{11}, x_{22} \neq 0 \right\}. \tag{5.13}$$

From Proposition 3.8, the expression of $\mathfrak{PM}$ is given as follows:

$$\mathfrak{PM} = \left\{ [g_\lambda.\langle , \rangle_0] \mid g_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}, \ \lambda > 0 \right\}. \tag{5.14}$$

For any $\lambda > 0$, one can see that

$$g_\lambda^{-1}(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))g_\lambda = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}). \tag{5.15}$$

This is an easy observation, but very important to get the following lemma.

Lemma 5.1. *Let $\mathfrak{g} = \mathfrak{r}_3$. Then the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is of cohomogeneity one, and all orbits are isometrically congruent to each other.*

Proof. In order to prove the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is of cohomogeneity one, it is enough to show that the orbit through $\langle , \rangle_0$ is of codimension one. From (5.13), it is easy to see that

$$\dim \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) = 5, \quad \dim(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \cap \mathrm{O}(3)) = 0. \tag{5.16}$$

Therefore $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle_0$ has dimension 5. This completes the proof, since the ambient space $\mathrm{GL}_3(\mathbb{R})/\mathrm{O}(3)$ has dimension 6.

Next we prove that all orbits are isometrically congruent to each other. Take any $\langle , \rangle$ and $\langle , \rangle'$. By Proposition 3.8, there exist $\lambda, \lambda' > 0$ such that

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g_\lambda.\langle , \rangle_0), \tag{5.17}$$

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle' = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g_{\lambda'}.\langle , \rangle_0). \tag{5.18}$$

We put $\mu := \lambda'/\lambda > 0$, and take $g_\mu \in \mathrm{GL}_3(\mathbb{R})$. Then (5.15) yields that

$$\begin{aligned} g_\mu \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle &= g_\mu \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g_\lambda.\langle , \rangle_0) \\ &= g_\mu (g_\mu^{-1} \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) g_\mu).(g_\lambda.\langle , \rangle_0) \\ &= \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g_{\lambda'}.\langle , \rangle_0) \\ &= \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle'. \end{aligned} \tag{5.19}$$

Thus g_μ maps the first orbit onto the second one, which completes the proof. $\square$

We refer to [12] for actions all of whose orbits are isometrically congruent to each other. Our idea of the proof of Lemma 5.1 comes from the arguments in [12].

Proposition 5.2. *Let $\mathfrak{g} = \mathfrak{r}_3$. Then the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\mathrm{GL}_3(\mathbb{R})/\mathrm{O}(3)$ has no minimal orbits.*

Proof. Consider the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\mathrm{GL}_3(\mathbb{R})/\mathrm{O}(3)$. From Lemma 5.1, all orbits are isometrically congruent to each other. Thus it is sufficient to prove that the orbit through the origin $\langle , \rangle_0$ is not minimal. We calculate the mean curvature vector of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle_0$. One can see from (3.12) that

$$\mathfrak{u}' := \mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \right\}, \tag{5.20}$$

$$\mathfrak{m}' := \mathrm{d}\pi_e(\mathfrak{u}') = \left\{ \begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{21} & x_{22} & x_{32} \\ x_{31} & x_{32} & x_{22} \end{pmatrix} \right\}. \tag{5.21}$$

Let us denote by E_{ij} the matrix whose (i,j)-entry is 1 and others are 0. We define a basis $\{X_1, \ldots, X_5\}$ of $\mathfrak{u}'$ by

$$\begin{aligned} X_1 &:= E_{11}, \quad X_2 := (1/\sqrt{2})(E_{22} + E_{33}), \\ X_3 &:= \sqrt{2}E_{21}, \quad X_4 := \sqrt{2}E_{31}, \quad X_5 := \sqrt{2}E_{32}. \end{aligned} \tag{5.22}$$

Furthermore we put

$$X'_i := (X_i)^*_o = (1/2)(X_i + {}^tX_i), \quad A := (1/\sqrt{2})(E_{22} - E_{33}). \tag{5.23}$$

Then $\{X'_1, \ldots, X'_5\}$ is an orthonormal basis of $\mathfrak{m}'$, and $\{A\}$ is an orthonormal basis of $\mathfrak{m} \ominus \mathfrak{m}'$. Recall that the mean curvature vector H is given by

$$H = -(1/5) \sum h(X'_i, X'_i), \quad \langle h(X'_i, X'_i), A \rangle = \langle [A, X_i]^*_o, (X_i)^*_o \rangle. \tag{5.24}$$

The bracket products $[A, X_i]$ satisfy

$$[A, X_1] = [A, X_2] = 0,\ [A, X_3] = E_{21},\ [A, X_4] = -E_{31},\ [A, X_5] = -2E_{32}. \tag{5.25}$$

Therefore, one has

$$\begin{aligned} \langle [A, X_3]^*_o, (X_3)^*_o \rangle &= \langle (1/2)(E_{21} + E_{12}), (\sqrt{2}/2)(E_{21} + E_{12}) \rangle = \sqrt{2}/2, \\ \langle [A, X_4]^*_o, (X_4)^*_o \rangle &= \langle (1/2)(-E_{31} - E_{13}), (\sqrt{2}/2)(E_{31} + E_{13}) \rangle = -\sqrt{2}/2, \\ \langle [A, X_5]^*_o, (X_5)^*_o \rangle &= \langle (-E_{32} - E_{23}), (\sqrt{2}/2)(E_{32} + E_{23}) \rangle = -\sqrt{2}. \end{aligned} \tag{5.26}$$

This yields that

$$H = (\sqrt{2}/5)A \neq 0. \tag{5.27}$$

Therefore, $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle, \rangle_0$ is not minimal, which completes the proof. $\square$

5.3. **Case of $\mathfrak{g} = \mathfrak{r}_{3,a}$ $(-1 \leq a < 1)$.** In this subsection, we study the case of $\mathfrak{g} = \mathfrak{r}_{3,a}$. Throughout this subsection, we fix a satisfying $-1 \leq a < 1$. Recall that, from Lemma 3.6, one has

$$\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix} \mid x_{11}, x_{21}, x_{22}, x_{31}, x_{33} \in \mathbb{R} \right\}. \tag{5.28}$$

The expression of $\mathfrak{PM}$ is given in Proposition 3.9 as follows:

$$\mathfrak{PM} = \left\{ [g_\lambda.\langle, \rangle_0] \mid g_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix},\ \lambda \in \mathbb{R} \right\}. \tag{5.29}$$

Proposition 5.3. *Let $\mathfrak{g} = \mathfrak{r}_{3,a}$. Then, we have*

(1) *The action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is of cohomogeneity one, and all orbits are hypersurfaces.*
(2) *$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle, \rangle_0$ is the unique minimal orbit.*

Proof. Take any $\langle,\rangle$. In order to prove (1), we show that $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle,\rangle$ is a hypersurface, that is, has dimension 5. From the expression of $\mathfrak{PM}$, there exists $\lambda \in \mathbb{R}$ such that

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle,\rangle = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g_\lambda.\langle,\rangle_0). \tag{5.30}$$

Let us define

$$U' := g_\lambda^{-1}(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))g_\lambda. \tag{5.31}$$

Then, since g_λ^{-1} gives an isometry, one has an isometric congruence

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g_\lambda.\langle,\rangle_0) \cong U'.\langle,\rangle_0. \tag{5.32}$$

Let $\mathfrak{u}'$ be the Lie algebra of U'. From the expression of $\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g})$, one can directly calculate

$$\mathfrak{u}' = g_\lambda^{-1}(\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}))g_\lambda = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & -\lambda(x_{22} - x_{33}) & x_{33} \end{pmatrix} \right\}. \tag{5.33}$$

Thus it is easy to check that

$$\dim \mathfrak{u}' = 5, \quad \dim(\mathfrak{u}' \cap \mathfrak{o}(3)) = 0. \tag{5.34}$$

Therefore $U'.\langle,\rangle_0$ has dimension 5, which completes the proof of (1).

In order to prove (2), we have only to show that $U'.\langle,\rangle$ is minimal if and only if $\lambda = 0$. From (5.33), one can see that

$$\mathfrak{m}' := \mathrm{d}\pi_e(\mathfrak{u}') = \left\{ \begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{21} & x_{22} & (-\lambda/2)(x_{22} - x_{33}) \\ x_{31} & (-\lambda/2)(x_{22} - x_{33}) & x_{33} \end{pmatrix} \right\}. \tag{5.35}$$

We define a basis $\{X_1, \ldots, X_5\}$ of $\mathfrak{u}'$ by

$$\begin{aligned} &X_1 := E_{11}, \quad X_2 := (1/\sqrt{2})(E_{22} + E_{33}), \\ &X_3 := (1/\sqrt{2(1+\lambda^2)})\left(E_{22} - E_{33} - 2\lambda E_{32}\right), \\ &X_4 := \sqrt{2}E_{21}, \quad X_5 := \sqrt{2}E_{31}. \end{aligned} \tag{5.36}$$

Let us put

$$X_i' := (X_i)_o^* = (1/2)(X_i + {}^tX_i). \tag{5.37}$$

Then $\{X_1', \ldots, X_5'\}$ is an orthonormal basis of $\mathfrak{m}'$. Furthermore, define

$$A := (1/\sqrt{2(1+\lambda^2)})\left(-\lambda E_{22} + \lambda E_{33} - 2E_{32}\right), \quad A' := (A)_o^*. \tag{5.38}$$

Then $\{A'\}$ is an orthonormal basis of $\mathfrak{m} \ominus \mathfrak{m}'$. Recall that the mean curvature vector H is given by

$$H = -(1/5)\sum h(X_i', X_i'), \quad \langle h(X_i', X_i'), A'\rangle = \langle [A, X_i]_o^*, (X_i)_o^*\rangle. \tag{5.39}$$

The bracket products $[A, X_i]$ satisfy

$$\begin{aligned} &[A, X_1] = [A, X_2] = 0, \quad [A, X_3] = -2E_{32}, \\ &[A, X_4] = -(1/\sqrt{1+\lambda^2})(\lambda E_{21} + 2E_{31}), \quad [A, X_5] = (\lambda/\sqrt{1+\lambda^2})E_{31}. \end{aligned} \tag{5.40}$$

Hence, one has

$$\begin{aligned} \langle [A, X_3]^*_o, (X_3)^*_o \rangle &= 2\lambda/\sqrt{2(1+\lambda^2)}, \\ \langle [A, X_4]^*_o, (X_4)^*_o \rangle &= -\lambda/\sqrt{2(1+\lambda^2)}, \\ \langle [A, X_5]^*_o, (X_5)^*_o \rangle &= \lambda/\sqrt{2(1+\lambda^2)}. \end{aligned} \tag{5.41}$$

This yields that

$$H = -2\lambda/(5\sqrt{2(1+\lambda^2)})A'. \tag{5.42}$$

Therefore, $H = 0$ if and only if $\lambda = 0$. This completes the proof of (2). □

5.4. **Case of $\mathfrak{g} = \mathfrak{r}'_{3,a}$ $(a \geq 0)$.** In this subsection, we study the case of $\mathfrak{g} = \mathfrak{r}'_{3,a}$. Throughout this subsection, we fix a satisfying $a \geq 0$. Recall that, from Lemma 3.6, $\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g})$ is given by

$$\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & -x_{23} \\ x_{31} & x_{23} & x_{22} \end{pmatrix} \right\}. \tag{5.43}$$

The expression of $\mathfrak{PM}$ is given in Proposition 3.10 as follows:

$$\mathfrak{PM} = \left\{ [g_\lambda.\langle , \rangle_0] \mid g_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}, \ \lambda \geq 1 \right\}. \tag{5.44}$$

Proposition 5.4. *Let $\mathfrak{g} = \mathfrak{r}'_{3,a}$. Then, we have*

1. *The action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is of cohomogeneity one, and $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle_0$ is the unique singular orbit.*
2. *$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle_0$ is the unique minimal orbit.*

Proof. Take any $\langle , \rangle$. In order to prove (1), we calculate the dimensions of the orbits. From the expression of $\mathfrak{PM}$, there exists $\lambda \geq 1$ such that

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle = \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g_\lambda.\langle , \rangle_0). \tag{5.45}$$

Let us denote by

$$U' := g_\lambda^{-1}(\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}))g_\lambda. \tag{5.46}$$

Then, since g_λ^{-1} gives an isometry, one has an isometric congruence

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(g_\lambda.\langle , \rangle_0) \cong U'.\langle , \rangle_0. \tag{5.47}$$

Let $\mathfrak{u}'$ be the Lie algebra of U'. From the expression of $\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g})$, a direct calculation yields that

$$\mathfrak{u}' = g_\lambda^{-1}(\mathbb{R} \oplus \mathrm{Der}(\mathfrak{g}))g_\lambda = \left\{ \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ x_{31} & -\lambda^2 x_{23} & x_{33} \end{pmatrix} \right\}. \tag{5.48}$$

Then we have

$$\dim \mathfrak{u}' = 5, \quad \dim(\mathfrak{u}' \cap \mathfrak{o}(3)) = \begin{cases} 0 & (\text{for } \lambda > 1), \\ 1 & (\text{for } \lambda = 1). \end{cases} \tag{5.49}$$

This yields that the orbit corresponding to $\lambda = 1$ is the unique singular orbit, (which has codimension two). This completes the proof of (1).

We show (2). It is known that every singular orbit of a cohomogeneity one action is minimal (see [22]). Then we have only to show that $U'.\langle , \rangle$ is not minimal if $\lambda > 1$. From now on assume that $\lambda > 1$. From (5.48), one can see that

$$\mathfrak{m}' := \mathrm{d}\pi_e(\mathfrak{u}') = \left\{ \begin{pmatrix} x_{11} & x_{21} & x_{31} \\ x_{21} & x_{22} & ((1-\lambda^2)/2)x_{23} \\ x_{31} & ((1-\lambda^2)/2)x_{23} & x_{33} \end{pmatrix} \right\}. \tag{5.50}$$

We define a basis $\{X_1, \ldots, X_5\}$ of $\mathfrak{u}'$ by

$$\begin{aligned} &X_1 := E_{11}, \quad X_2 := (1/\sqrt{2})(E_{22} + E_{33}), \quad X_3 := \sqrt{2}E_{21}, \\ &X_4 := \sqrt{2}E_{31}, \quad X_5 := (\sqrt{2}/(1-\lambda^2))(E_{23} - \lambda^2 E_{32}). \end{aligned} \tag{5.51}$$

Furthermore we put

$$X_i' := (X_i)_o^* = (1/2)(X_i + {}^t X_i), \quad A := (1/\sqrt{2})(E_{22} - E_{33}). \tag{5.52}$$

Then $\{X_1', \ldots, X_5'\}$ is an orthonormal basis of $\mathfrak{m}'$, and $\{A\}$ is an orthonormal basis of $\mathfrak{m} \ominus \mathfrak{m}'$. Recall that the mean curvature vector H is given by

$$H = -(1/5) \sum h(X_i', X_i'), \quad \langle h(X_i', X_i'), A \rangle = \langle [A, X_i]_o^*, (X_i)_o^* \rangle. \tag{5.53}$$

The bracket products $[A, X_i]$ satisfy

$$\begin{aligned} &[A, X_1] = [A, X_2] = 0, \quad [A, X_3] = E_{21}, \quad [A, X_4] = -E_{31}, \\ &[A, X_5] = (2/(1-\lambda^2))(E_{23} + \lambda^2 E_{32}). \end{aligned} \tag{5.54}$$

Hence, one has

$$\begin{aligned} \langle [A, X_3]_o^*, (X_3)_o^* \rangle &= 1/\sqrt{2}, \\ \langle [A, X_4]_o^*, (X_4)_o^* \rangle &= -1/\sqrt{2}, \\ \langle [A, X_5]_o^*, (X_5)_o^* \rangle &= \sqrt{2}(1+\lambda^2)/(1-\lambda^2). \end{aligned} \tag{5.55}$$

This yields that

$$H = -\sqrt{2}(1+\lambda^2)/(5(1-\lambda^2))A \neq 0. \tag{5.56}$$

which completes the proof. □

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